A circular loop located on $x^2 + y^2 = 9$, $z = 0$ carries a direct current of 10 A along $\mathbf{a}_\phi$. Determine $\mathbf{H}$ at (0, 0, 4) and (0, 0, −4).

Solution:

Consider the circular loop shown in Figure 7.8(a). The magnetic field intensity $d\mathbf{H}$ at point $P(0, 0, h)$ contributed by current element $I\,d\mathbf{l}$ is given by Biot–Savart's law:

$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

where $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$, $\mathbf{R} = (0, 0, h) - (x, y, 0) = -\rho\mathbf{a}_\rho + h\mathbf{a}_z$, and

$$d\mathbf{l} \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\, d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z$$

Hence,

$$d\mathbf{H} = \frac{I}{4\pi[\rho^2 + h^2]^{3/2}} (\rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z) = dH_\rho\, \mathbf{a}_\rho + dH_z\, \mathbf{a}_z$$

By symmetry, the contributions along $\mathbf{a}_\rho$ add up to zero because the radial components produced by pairs of current element 180° apart cancel. This may also be shown mathematically by writing $\mathbf{a}_\rho$ in rectangular coordinate systems (i.e., $\mathbf{a}_\rho = \cos\phi\, \mathbf{a}_x + \sin\phi\, \mathbf{a}_y$).

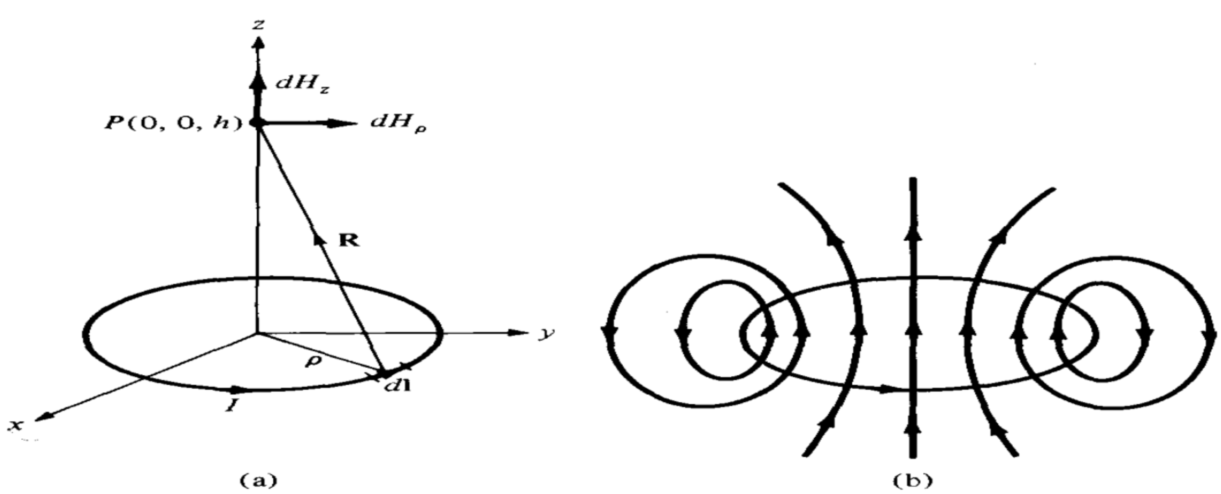

Figure 7.8 For Example 7.3: **(a)** circular current loop, **(b)** flux lines due to the current loop.

Integrating $\cos\phi$ or $\sin\phi$ over $0 \le \phi \le 2\pi$ gives zero, thereby showing that $\mathbf{H}_\rho = 0$. Thus

$$\mathbf{H} = \int dH_z\,\mathbf{a}_z = \int_0^{2\pi} \frac{I\rho^2\,d\phi\,\mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}} = \frac{I\rho^2 2\pi\mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}}$$

or

$$\mathbf{H} = \frac{I\rho^2\mathbf{a}_z}{2[\rho^2 + h^2]^{3/2}}$$

(a) Substituting $I = 10\,A$, $\rho = 3$, $h = 4$ gives

$$\mathbf{H}(0, 0, 4) = \frac{10\,(3)^2\,\mathbf{a}_z}{2[9 + 16]^{3/2}} = 0.36\mathbf{a}_z\text{ A/m}$$

A current filament of $3\mathbf{a}_x$ A lies along the x axis. Find $\mathbf{H}$ in cartesian components at $P(-1, 3, 2)$: We use the Biot-Savart law,

$$\mathbf{H} = \int \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

where $Id\mathbf{L} = 3dx\mathbf{a}_x$, $\mathbf{a}_R = [-(1+x)\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z]/R$, and $R = \sqrt{x^2 + 2x + 14}$. Thus

$$\mathbf{H}_P = \int_{-\infty}^{\infty} \frac{3dx\mathbf{a}_x \times [-(1+x)\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z]}{4\pi(x^2 + 2x + 14)^{3/2}} = \int_{-\infty}^{\infty} \frac{(9\mathbf{a}_z - 6\mathbf{a}_y)\,dx}{4\pi(x^2 + 2x + 14)^{3/2}}$$

$$= \frac{(9\mathbf{a}_z - 6\mathbf{a}_y)(x+1)}{4\pi(13)\sqrt{x^2 + 2x + 14}}\Bigg|_{-\infty}^{\infty} = \frac{2(9\mathbf{a}_z - 6\mathbf{a}_y)}{4\pi(13)} = \underline{0.110\mathbf{a}_z - 0.073\mathbf{a}_y\text{ A/m}}$$

Assume that there is a region with cylindrical symmetry in which the conductivity is given by $\sigma = 1.5e^{-150\rho}$ kS/m. An electric field of 30 $\mathbf{a}_z$ V/m is present.

a) Find **J**: Use

$$\mathbf{J} = \sigma\mathbf{E} = \underline{45e^{-150\rho}\,\mathbf{a}_z\ \text{kA/m}^2}$$

b) Find the total current crossing the surface $\rho < \rho_0$, $z = 0$, all ϕ:

$$I = \int\int \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\rho_0} 45e^{-150\rho}\rho\,d\rho\,d\phi = \frac{2\pi(45)}{(150)^2}e^{-150\rho}\left[-150\rho - 1\right]\Big|_0^{\rho_0}\ \text{kA}$$
$$= \underline{12.6\left[1 - (1 + 150\rho_0)e^{-150\rho_0}\right]\ \text{A}}$$

c) Make use of Ampere's circuital law to find **H**: Symmetry suggests that **H** will be ϕ-directed only, and so we consider a circular path of integration, centered on and perpendicular to the z axis. Ampere's law becomes: $2\pi\rho H_\phi = I_{encl}$, where I_{encl} is the current found in part b, except with ρ_0 replaced by the variable, ρ. We obtain

$$H_\phi = \underline{\frac{2.00}{\rho}\left[1 - (1 + 150\rho)e^{-150\rho}\right]\ \text{A/m}}$$

The magnetic field intensity is given in the square region $x = 0$, $0.5 < y < 1$, $1 < z < 1.5$ by $\mathbf{H} = z^2\mathbf{a}_x + x^3\mathbf{a}_y + y^4\mathbf{a}_z$ A/m.

a) evaluate $\oint \mathbf{H}\cdot d\mathbf{L}$ about the perimeter of the square region: Using $d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$, and using the given field, we find, in the $x = 0$ plane:

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int_{.5}^{1} 0\,dy + \int_1^{1.5}(1)^4\,dz + \int_1^{.5} 0\,dy + \int_{1.5}^{1}(.5)^4\,dz = \underline{0.46875}$$

b) Find $\nabla\times\mathbf{H}$:

$$\nabla\times\mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$
$$= \underline{4y^3\mathbf{a}_x + 2z\mathbf{a}_y + 3x^2\mathbf{a}_z}$$

c) Calculate $(\nabla\times\mathbf{H})_x$ at the center of the region: Here, $y = 0.75$ and so $(\nabla\times\mathbf{H})_x = 4(.75)^3 = \underline{1.68750}$.

d) Does $(\nabla\times\mathbf{H})_x = [\oint\mathbf{H}\cdot d\mathbf{L}]/\text{Area Enclosed}$? Using the part a result, $[\oint\mathbf{H}\cdot d\mathbf{L}]/\text{Area Enclosed} = 0.46875/0.25 = \underline{1.8750}$, which is off the value found in part c. Answer: No. Reason: the limit of the area shrinking to zero must be taken before the results will be equal.

Evaluate both sides of Stokes' theorem for the field $\mathbf{G} = 10 \sin\theta\, \mathbf{a}_\phi$ and the surface $r = 3, 0 \leq \theta \leq 90^\circ$, $0 \leq \phi \leq 90^\circ$. Let the surface have the $\mathbf{a}_r$ direction: Stokes' theorem reads:

$$\oint_C \mathbf{G} \cdot d\mathbf{L} = \int\int_S (\nabla \times \mathbf{G}) \cdot \mathbf{n}\, da$$

Considering the given surface, the contour, C, that forms its perimeter consists of three joined arcs of radius 3 that sweep out 90° in the xy, xz, and zy planes. Their centers are at the origin. Of these three, only the arc in the xy plane (which lies along $\mathbf{a}_\phi$) is in the direction of $\mathbf{G}$; the other two (in the $-\mathbf{a}_\theta$ and $\mathbf{a}_\theta$ directions respectively) are perpendicular to it, and so will not contribute to the path integral. The left-hand side therefore consists of only the xy plane portion of the closed path, and evaluates as

$$\oint \mathbf{G} \cdot d\mathbf{L} = \int_0^{\pi/2} 10 \sin\theta\big|_{\pi/2}\, \mathbf{a}_\phi \cdot \mathbf{a}_\phi\, 3 \sin\theta\big|_{\pi/2}\, d\phi = \underline{15\pi}$$

To evaluate the right-hand side, we first find

$$\nabla \times \mathbf{G} = \frac{1}{r \sin\theta} \frac{d}{d\theta} \left[(\sin\theta) 10 \sin\theta \right] \mathbf{a}_r = \frac{20 \cos\theta}{r} \mathbf{a}_r$$

The surface over which we integrate this is the one-eighth spherical shell of radius 3 in the first octant, bounded by the three arcs described earlier. The right-hand side becomes

$$\int\int_S (\nabla \times \mathbf{G}) \cdot \mathbf{n}\, da = \int_0^{\pi/2} \int_0^{\pi/2} \frac{20 \cos\theta}{3} \mathbf{a}_r \cdot \mathbf{a}_r\, (3)^2 \sin\theta\, d\theta\, d\phi = \underline{15\pi}$$

It would appear that the theorem works.

Let $\mathbf{G} = 15r\,\mathbf{a}_\phi$.

a) Determine $\oint \mathbf{G}\cdot d\mathbf{L}$ for the circular path $r = 5, \theta = 25^\circ, 0 \le \phi \le 2\pi$:

$$\oint \mathbf{G}\cdot d\mathbf{L} = \int_0^{2\pi} 15(5)\mathbf{a}_\phi \cdot \mathbf{a}_\phi (5)\sin(25^\circ)\,d\phi = 2\pi(375)\sin(25^\circ) = \underline{995.8}$$

b) Evaluate $\int_S (\nabla\times\mathbf{G})\cdot d\mathbf{S}$ over the spherical cap $r = 5, 0 \le \theta \le 25^\circ, 0 \le \phi \le 2\pi$: When evaluating the curl of $\mathbf{G}$ using the formula in spherical coordinates, only one of the six terms survives:

$$\nabla\times\mathbf{G} = \frac{1}{r\sin\theta}\frac{\partial(G_\phi\sin\theta)}{\partial\theta}\mathbf{a}_r = \frac{1}{r\sin\theta}15r\cos\theta\,\mathbf{a}_r = 15\cot\theta\,\mathbf{a}_r$$

Then

$$\int_S (\nabla\times\mathbf{G})\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{25^\circ} 15\cot\theta\,\mathbf{a}_r\cdot\mathbf{a}_r\,(5)^2\sin\theta\,d\theta\,d\phi$$

$$= 2\pi\int_0^{25^\circ} 15\cos\theta(25)\,d\theta = 2\pi(15)(25)\sin(25^\circ) = \underline{995.8}$$

The magnetic field intensity is given in a certain region of space as

$$\mathbf{H} = \frac{x+2y}{z^2}\mathbf{a}_y + \frac{2}{z}\mathbf{a}_z \text{ A/m}$$

a) Find $\nabla\times\mathbf{H}$: For this field, the general curl expression in rectangular coordinates simplifies to

$$\nabla\times\mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x + \frac{\partial H_y}{\partial x}\mathbf{a}_z = \underline{\frac{2(x+2y)}{z^3}\mathbf{a}_x + \frac{1}{z^2}\mathbf{a}_z \text{ A/m}}$$

b) Find $\mathbf{J}$: This will be the answer of part a, since $\nabla\times\mathbf{H} = \mathbf{J}$.

c) Use $\mathbf{J}$ to find the total current passing through the surface $z = 4, 1 < x < 2, 3 < y < 5$, in the $\mathbf{a}_z$ direction: This will be

$$I = \int\int \mathbf{J}\big|_{z=4}\cdot\mathbf{a}_z\,dx\,dy = \int_3^5\int_1^2 \frac{1}{4^2}dx\,dy = \underline{1/8 \text{ A}}$$

d) Show that the same result is obtained using the other side of Stokes' theorem: We take $\oint \mathbf{H}\cdot d\mathbf{L}$ over the square path at $z = 4$ as defined in part c. This involves two integrals of the y component of $\mathbf{H}$ over the range $3 < y < 5$. Integrals over x, to complete the loop, do not exist since there is no x component of $\mathbf{H}$. We have

$$I = \oint \mathbf{H}\big|_{z=4}\cdot d\mathbf{L} = \int_3^5 \frac{2+2y}{16}dy + \int_5^3 \frac{1+2y}{16}dy = \frac{1}{8}(2) - \frac{1}{16}(2) = \underline{1/8 \text{ A}}$$

A radial field

$$\mathbf{H} = \frac{2.39 \times 10^6}{r} \cos\phi \mathbf{a}_r \text{ A/m}$$

exists in free space. Find the magnetic flux Φ crossing the surface defined by $-\pi/4 \le \phi \le \pi/4$, $0 \le z \le 1$ m. See Fig. 9-16.

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{3.00}{r} \cos\phi \mathbf{a}_r \quad \text{(T)}$$

$$\Phi = \int_0^1 \int_{-\pi/4}^{\pi/4} \left(\frac{3.00}{r}\cos\phi\right)\mathbf{a}_r \cdot r\,d\phi\,dz\,\mathbf{a}_r$$

$$= 4.24 \text{ Wb}$$

Since $\mathbf{B}$ is inversely proportional to r (as required by $\nabla \cdot \mathbf{B} = 0$), it makes no difference what radial distance is chosen, the total flux will be the same.

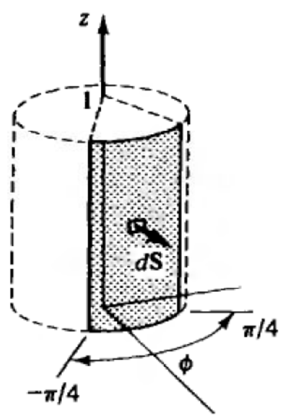

Fig. 9-16

In cylindrical coordinates, $\mathbf{B} = (2.0/r)\mathbf{a}_\phi$ (T). Determine the magnetic flux Φ crossing the plane surface defined by $0.5 \le r \le 2.5$ m and $0 \le z \le 2.0$ m.

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S}$$

$$= \int_0^{2.0} \int_{0.5}^{2.5} \frac{2.0}{r} \mathbf{a}_\phi \cdot dr\,dz\,\mathbf{a}_\phi$$

$$= 4.0\left(\ln\frac{2.5}{0.5}\right) = 6.44 \text{ Wb}$$